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Conformal invariance and critical spectrum of corner transfer matrices

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Abstract. The finite-size scaling spectra of Ising model corner transfer matrices and their generators are studied at criticality. The generators are diagonalised using fermion algebra. The low-lying eigenvalues, given by the zeros of Meixner polynomials, are equally spaced and collapse like $1/\log N$ for large N as predicted by conformal invariance. The asymptotics are obtained using a generalised Euler-Maclaurin summation formula. The shift in the largest eigenvalue is given analytically as $\pi c/6 \log N$ with central charge $c = \frac{1}{2}$. The spectrum generating functions, for both fixed and free boundary conditions, are expressed simply in terms of the $c = \frac{1}{2}$ Virasoro characters $\chi_\Delta(q)$ with modular parameter $q = \exp(-\pi/\log N)$ and conformal dimensions $\Delta = 0, \frac{1}{2}, \frac{1}{16}$.

1. Introduction

Recently much progress has been made in the theory of critical phenomena by generalising the notion of scale invariance of critical systems to include invariance under conformal and modular transformations (Belavin *et al* 1984, Cardy 1986a, 1987, 1988). In particular, the universality classes of critical two-dimensional lattice models and conformal field theories can be classified according to the value of the central charge c of the corresponding conformal Virasoro algebra. For $c < 1$, a complete classification has been obtained (Friedan *et al* 1984) yielding the unitary series with $c = 1 - 6/m(m+1)$ and $m = 3, 4, 5, \ldots$. In this case the critical exponents are quantised and only a finite number of scaling operators enter the theory. These theories are realised within statistical mechanics by the *A-D-E* lattice models (Andrews *et al* 1984, Huse 1984, Pasquier 1986, 1987a, b). The additional assumption of modular invariance on a torus allows the complete operator content of these theories to be elucidated (Cardy 1986a). More precisely, the finite-size scaling partition function at criticality is given as a modular invariant combination of the characters of the associated Virasoro algebra (Itzykson and Zuber 1986, Capelli *et al* 1987a, b).

The study of conformal invariance in other geometries and the effect of boundary conditions has been initiated by Cardy (1986b). It now appears that the operator content of the theory on an annulus plays a privileged role (Cardy 1990) and that this is intimately connected with the very special properties of corner transfer matrices (CTMs) introduced by Baxter (1976, 1982). Perhaps the most striking evidence for this

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is the mysterious occurrence of Virasoro characters in the spectra of CTMs of lattice models off criticality (Date *et al* 1987, 1989, Kuniba and Yajima 1987, Saleur and Bauer 1989). This naturally leads to the possibility of extending the mathematical machinery of Virasoro algebras to integrable non-critical models (Thacker 1986, Itoyama and Thacker 1986, 1988, 1989). Apparently, in this special case, the Virasoro symmetry is connected with integrability and the Yang-Baxter equation and is not a consequence of conformal invariance.

Following Peschel and Truong (Peschel and Truong 1987, Truong and Peschel 1988a, b, Peschel 1988), we pursue here the study of the spectra of lattice model CTMs at criticality. In this case the occurrence of the Virasoro algebra is expected as a consequence of the conformal invariance of the critical system. Indeed, on the basis of conformal invariance, Peschel and Truong have established two important facts. First, for a corner of large but finite size N and opening angle θ, the low-lying eigenvalues are equally spaced with level splitting given by

$$\varepsilon = \pi\theta/\log N. \tag{1.1}$$

Second, the universal finite-size correction to the largest eigenvalue or groundstate is given by

$$\pi c/24 \log N \tag{1.2}$$

where c is the central charge.

In this paper, we extend the Ising model results of Peschel and Truong in two directions. First, we are able to calculate the universal finite-size corrections for the CTM generators analytically. This is achieved using fermion techniques developed in Davies (1988, 1989) and a generalised Euler-Maclaurin summation formula (Lyness and Ninham 1966). We find the well known Ising result

$$c = \tfrac{1}{2}. \tag{1.3}$$

Second, we explore the dependence of the critical CTM spectra on boundary conditions. Specifically, we obtain the low-lying eigenvalues for both fixed and free boundary conditions. The generating functions of these eigenvalues are expressed simply in terms of the $c = \frac{1}{2}$ Virasoro characters $\chi_\Delta(q)$ with modular parameter $q = \exp(-\pi/\log N)$ and conformal dimensions $\Delta = 0, \frac{1}{2}, \frac{1}{16}$. We treat only the Ising model here for the technical convenience that there is a fermion representation. Nevertheless, the clear general picture that emerges should apply to lattice models in general and to the *A-D-E* models in particular.

2. Corner transfer matrices and their generators

The basic CTM for a lattice divided at a given centre spin s_1 and with $N = 4$ spins along an edge is shown in figures 1 and 2. The difference between the two is in the choice of boundary conditions: in figure 1 all boundary spins are free, in figure 2 they are all fixed. In this paper we are interested in the critical Ising model for which (the J_i are normalised by kT)

$$\sinh 2J_1 \sinh 2J_2 = 1. \tag{2.1}$$

A standard parametrisation of the interaction coefficients in this case is

$$\begin{aligned} \sinh 2\hat{J}_1 &= \sinh 2J_2 = \tan 2u \\ \cosh 2\hat{J}_1 &= \cosh 2J_2 = \sec 2u. \end{aligned} \tag{2.2}$$

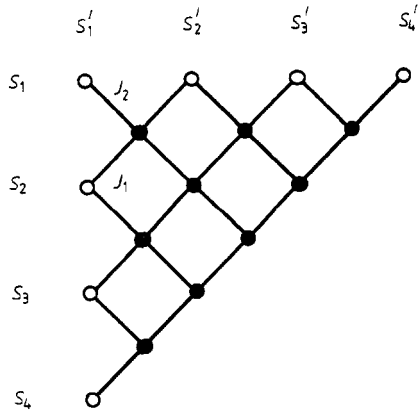

Figure 1. Corner transfer matrix for free boundary conditions. Circles are spins, full circles are summed out.

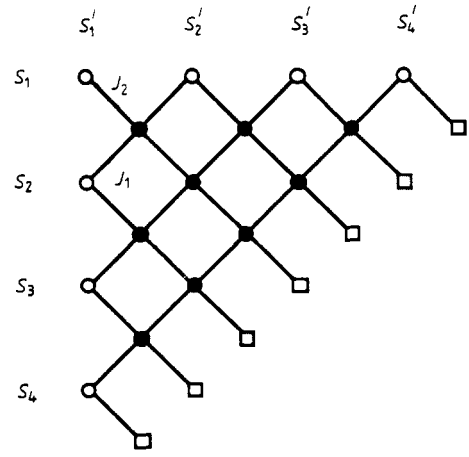

Figure 2. Corner transfer matrix for fixed boundary conditions. Circles and squares are spins, full circles are summed out and squares are fixed boundary spins.

Here we have introduced dual variables in the usual manner:

$$\tanh \hat{J}_i = \exp(-2J_i). \tag{2.3}$$

In (2.2), u is the spectral parameter. It is also a measure of the anisotropy of the system: in the extreme anisotropic limit $A(u)$ approaches a multiple of the identity matrix. Rotation of the lattice through 90°, which interchanges J_1 with J_2, is achieved by $u \to \lambda - u$, where $\lambda = \pi/4$ is the crossing parameter; the physical region is $0 < u < \lambda$.

In the general case, there will be N spins along an edge: for fixed boundary conditions we envisage a line of fixed spins on the $(N+1)$th line. The interaction between two neighbouring spins defines an operator represented by a 2×2 matrix, the entries of which are the Boltzmann weights $\exp(\pm J_i)$. To write these explicitly in terms of Pauli matrices σ_j^α $(\alpha = x, y, z)$ we choose the non-standard representation

$$\sigma_j^x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \sigma_j^y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \qquad \sigma_j^z = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}. \tag{2.4}$$

(This differs from the standard representation only via a 90° rotation about the y axis.) The operator for an interaction of strength J_1 between two spins on the jth line is

$$\begin{aligned} X_{2j-1} &= (2\sinh 2J_1)^{1/2}\exp(-\hat{J}_1\sigma_j^z) \\ &= \rho_1\left(I+\frac{\sin u}{\sin(\lambda-u)}U_{2j-1}\right) \\ U_{2j-1} &= 2^{-1/2}(I-\sigma_j^z). \end{aligned} \tag{2.5}$$

Here ρ_1 is an unimportant normalisation factor. Similarly, the operator for an interaction of strength J_2 between two spins on the jth and $(j+1)$th lines is

$$\begin{aligned} X_{2j} &= \exp(J_2\sigma_j^x\sigma_{j+1}^x) \\ &= \rho_2\left(I+\frac{\sin u}{\sin(\lambda-u)}U_{2j}\right) \\ U_{2j} &= 2^{-1/2}(I+\sigma_j^x\sigma_{j+1}^x). \end{aligned} \tag{2.6}$$

Finally, the operator for an interaction of strength J_2 between a spin on the Nth line and a fixed spin on the $(N+1)$th boundary line is

$$\begin{aligned} X_{2N} &= \exp(J_2\sigma_N^x) \\ &= \rho_2\left(I+\frac{\sin u}{\sin(\lambda-u)}U_{2N}\right) \\ U_{2N} &= 2^{-1/2}(I+\sigma_N^x). \end{aligned} \tag{2.7}$$

Here the convention is that the boundary spin is fixed so that it is an eigenstate of σ_{N+1}^x with eigenvalue $+1$.

Clearly the CTM is the product of such operators, one for each interaction, in the appropriate order. For free boundaries, the product is

$$\begin{aligned} A(u) = {} & X_{2N-1}X_{2N-2}(X_{2N-1}X_{2N-3})(X_{2N-2}X_{2N-4})(X_{2N-1}X_{2N-3}X_{2N-5}) \\ & \ldots X_2\ldots(X_{2N-5}X_{2N-3}X_{2N-1})(X_{2N-4}X_{2N-2})(X_{2N-3}X_{2N-1})X_{2N-2}X_{2N-1}. \end{aligned} \tag{2.8}$$

The operator at the centre of this product is X_2 and the factors X_k in the product are symmetrically disposed about this centre factor. This is important: it means that the inverse of $A(u)$ is the same product with each factor X_k replaced by its inverse X_k^{-1}. For fixed boundaries, $A(u)$ is obtained by the trivial modification

$$\begin{aligned} A(u) = {} & X_{2N}X_{2N-1}(X_{2N}X_{2N-2})(X_{2N-1}X_{2N-3})(X_{2N}X_{2N-2}X_{2N-4}) \\ & \ldots X_2\ldots(X_{2N-4}X_{2N-2}X_{2N})(X_{2N-3}X_{2N-1})(X_{2N-2}X_{2N})X_{2N-1}X_{2N}. \end{aligned} \tag{2.9}$$

For a non-critical system, and in the thermodynamic limit ($N\to\infty$), Baxter (1982) has shown that there is an operator H called the generator such that $A_n(u)=\exp(-uH)$. At criticality the situation is more delicate because the spectrum collapses. However we shall see that there is a generator H which contains much useful information in this case also. Expanding $A(u)$ about $u=0$ the first non-trivial term is the linear one. To this order non-commutation of the operators is no problem and a simple computation gives H. Using the definitions

$$A_n(u)=\prod_i(I+uU_i)\approx I-uH \tag{2.10}$$

we find, in the case of free boundaries,

$$H_0 = -\sqrt{2}\sum_{j=1}^{2N-1}(j-1)U_j \tag{2.11}$$

and for fixed boundaries

$$H_{\pm} = -\sqrt{2}\sum_{j=1}^{2N}(j-1)U_j. \tag{2.12}$$

The general structure for the CTM and its generator H is generic for many solvable models, with the operators U_j forming a Temperley-Lieb algebra (Temperley and Lieb 1971, Baxter 1982), explicitly:

$$\begin{aligned} &U_j^2 = Q^{1/2}U_j \\ &U_jU_{j\pm1}U_j = U_j \\ &U_iU_j = U_jU_i \qquad |i-j|>1. \end{aligned} \tag{2.13}$$

For Potts models the parameter Q is identified with the number of states; for the Ising model $Q=2$. An important property which we need, which follows immediately from our definitions and the Temperley-Lieb algebra, is the inversion identity (Davies 1988)

$$A(u)A(-u) = (2\cot 2u)^{N_1}I \tag{2.14}$$

where N_1 is the number of interactions of type J_1. To prove this result, we note that there is a similar identity for each of the operators X_j, namely

$$X_{2j-1}(u)X_{2j-1}(-u) = (2\cot 2u)I \qquad X_{2j}(u)X_{2j}(-u) = I \tag{2.15}$$

from which (2.14) follows immediately. For the Ising model it is natural to use fermion operators and Clifford algebras to represent solutions of the Ising model in an elegant and simple form. This is the subject of the next two sections.

3. Diagonalisation of generators

It is convenient to introduce fermion operators first, via the Jordan-Wigner transformation:

$$\begin{aligned} f_j^+ &= P_{j-1}(\sigma_j^x + \mathrm{i}\sigma_j^y)/2 \\ f_j &= P_{j-1}(\sigma_j^x - \mathrm{i}\sigma_j^y)/2 \end{aligned} \tag{3.1}$$

where the P_j are defined by

$$P_j = \prod_{k=1}^{j}(-\sigma_k^z). \tag{3.2}$$

The operators f_j^+, f_j satisfy the usual anticommutation relations for fermions. More important for our immediate purpose are the operators

$$\begin{aligned} \Gamma_{2j-1} &= (f_j^+ + f_j) = P_{j-1}\sigma_j^x \\ \Gamma_{2j} &= -\mathrm{i}(f_j^+ - f_j) = P_{j-1}\sigma_j^y \end{aligned} \tag{3.3}$$

together with the operator $\Gamma_{2N+1} = P_N$ which may be written as the product

$$\Gamma_{2N+1} = \mathrm{i}^N\Gamma_1\Gamma_2\ldots\Gamma_{2N-1}\Gamma_{2N}. \tag{3.4}$$

These operators satisfy the commutation relations for generators of a Clifford algebra:

$$\Gamma_j\Gamma_k + \Gamma_k\Gamma_j = 2\delta_{jk}. \tag{3.5}$$

It is important for our analysis to notice that there are two natural ways in which these operators may be related to fermions. One is the way in which they were constructed, and then Γ_{2N+1} is the parity operator which commutes with everything for the case of free boundary conditions. However, for fixed boundary conditions the CTM does not commute with Γ_{2N+1}, so this choice is inappropriate. Since it does commute with Γ_1, it is natural to use the operators $\Gamma_2, \Gamma_3, \ldots, \Gamma_{2N}, \Gamma_{2N+1}$, to generate the Clifford algebra in this case, with sums and differences of the pairs $(\Gamma_2, \Gamma_3), \ldots, (\Gamma_{2N}, \Gamma_{2N+1})$ as fermion creation and annihilation operators. Now the further relation $\Gamma_1 = \mathrm{i}^N\Gamma_2\Gamma_3\ldots\Gamma_{2N}\Gamma_{2N+1}$ means that the centre spin is determined by the parity operator Γ_1. This relabelling of the Clifford generators is the 'order-disorder' transformation used in Davies (1988). It enables us to treat both sets of boundary conditions in a parallel way.

The operators X_{2j-1}, X_{2j}, have a simple interpretation as spinor rotations, namely

$$\begin{aligned} X_{2j-1} &= (2\sinh 2J_1)^{1/2}\exp(\mathrm{i}\hat{J}_1\Gamma_{2j-1}\Gamma_{2j}) \\ X_{2j} &= \exp(\mathrm{i}J_2\Gamma_{2j}\Gamma_{2j+1}). \end{aligned} \tag{3.6}$$

This also expresses the operators H as sums of such quadratic factors $\Gamma_j\Gamma_{j+1}$. We want to bring H to diagonal form in fermions by a Bogoliubov-Valatin (Bogoliubov 1959, Valatin 1961) transformation, i.e. we seek a canonical transformation to new fermion operators G_j^+, G_j, according to which the new operators are linear combination of the old, so that H takes the form of a sum of commuting operators. Most of the technical details may be found in Davies (1988): here we only give what is needed for our immediate purpose. The appropriate diagonal form is

$$H = \sum_{j=1}^{N} 4\varepsilon_j G_j^+ G_j. \tag{3.7}$$

We must explain the factor 4. The partition function of a complete lattice is obtained from the product of four corners: $A(u)A(\pi/4-u)A(u)A(\pi/4-u) = \exp(\pi H/2)$. We want to relate the low-lying eigenvalues through the correspondence $\lambda_n(u) = \exp(-\theta\varepsilon_j)$, where θ is the opening angle, and $\theta = 2\pi$ corresponds to $u = \pi/2$. Thus, with this factor inserted, all conformal results will be normalised by the number of radians in a complete planar lattice.

Denote the new Clifford generators corresponding to the new fermion operators by

$$\Gamma'_{2j-1} = (G_j^+ + G_j) \qquad \Gamma'_{2j} = -\mathrm{i}(G_j^+ - G_j). \tag{3.8}$$

The transformation will be canonical, i.e. the new generators will satisfy (3.5), if they are related to the original generators by a real orthogonal linear transformation:

$$\Gamma'_i = \sum P_{ij}\Gamma_j \qquad (1 \leqslant i \leqslant 2N). \tag{3.9}$$

We have omitted the summation limits in this equation: in the case of free boundary conditions we have $1 \leqslant j \leqslant 2N$, for fixed boundary conditions we have $2 \leqslant j \leqslant 2N+1$. Determination of P_{ij} and ε_j is now an eigenvalue problem for a matrix of dimension $2N$, found by requiring the commutation relation (Davies 1988, equation (3.6))

$$[H, G_j^+] = 4\varepsilon_j G_j^+. \tag{3.10}$$

It is illuminating to express H_0, $H_\pm$ in fermion operators. We find

$$H_0 = \sum_{j=1}^{N} 4(j-1)(f_j^\dagger f_j - \tfrac{1}{2}) - \sum_{j=1}^{N-1} (2j-1)(f_j^\dagger f_{j+1} + f_{j+1}^\dagger f_j + f_j^\dagger f_{j+1}^\dagger - f_j f_{j+1}) \tag{3.11}$$

which is the critical limit of the high-temperature case (Davies 1988, equation (3.4)) and

$$H_\pm = \sum_{j=1}^{N} 2(2j-1)(f_j^\dagger f_j - \tfrac{1}{2}) - \sum_{j=1}^{N-1} 2j(f_j^\dagger f_{j+1} + f_{j+1}^\dagger f_j + f_j^\dagger f_{j+1}^\dagger - f_j f_{j+1}) \tag{3.12}$$

which is the critical limit of the low-temperature case (Davies 1988, equation (5.6)). The difference is brought about by the different choice of generators for the algebra. In either case a linear transformation (Davies 1988, equation (3.7)) leads to an eigenvalue problem for a real symmetric tridiagonal matrix $\mathscr{H}_0$ ($\mathscr{H}_\pm$) of dimension $2N$ (Davies 1988, equations (3.8), (5.8)) and we obtain a real orthogonal transformation P for use in (3.9). The matrices $\mathscr{H}_0$, $\mathscr{H}_\pm$, for the two cases are bipartite and have the following structure:

$$\begin{aligned} (\mathscr{H}_0)_{ij} &= \tfrac{1}{2}(i-1)\delta_{i,j+1} + \tfrac{1}{2}(j-1)\delta_{i+1,j} && (1 \leqslant i, j \leqslant 2N-1) \\ (\mathscr{H}_\pm)_{ij} &= \tfrac{1}{2} i \delta_{i,j+1} + \tfrac{1}{2} j \delta_{i+1,j} && (1 \leqslant i, j \leqslant 2N-1). \end{aligned} \tag{3.13}$$

For example, in the case $N = 2$:

$$\mathscr{H}_0 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad \mathscr{H}_\pm = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix}. \tag{3.14}$$

The only difference is that $\mathscr{H}_0$ has a border of zeros added to $\mathscr{H}_\pm$ as its first row and column, with the last row and column of $\mathscr{H}_\pm$ omitted. The border of zeros gives a zero eigenvalue with eigenvector $(1, 0, 0, 0, \ldots)$: this informs us that σ_1^x commutes with H_0. The rest of the spectrum, in both cases, is found by diagonalising the same matrix: for free boundary conditions it has $2N-1$ rows and columns and for fixed boundary conditions $2N$. Because the matrices are symmetric and bipartite, the eigenvalues occur in real pairs of opposite sign.

4. Diagonalisation of corner transfer matrices

We also need to know the structure of the diagonal form of the CTMs in fermion operators. We seek a canonical transformation under which the CTMs take the form of a product of commuting operators:

$$\begin{aligned} A(u) &= \Lambda_{\max} \prod_{j=1}^{N} [1 + (\lambda_j - 1) G_j^\dagger G_j] \\ &= \Lambda_{\max} \exp\Big(\sum_{j=1}^{N} (\log \lambda_j) G_j^\dagger G_j \Big). \end{aligned} \tag{4.1}$$

We make the following observations before proceeding further.

(i) The CTMs commute with $\Gamma_1 = \sigma_1^x$ so they are diagonal in the first spin variable, falling into two blocks labelled by the centre spin.

(ii) In the case of free boundary conditions, the CTM also commutes with Γ_{2N+1}, the operator which detects the parity of the states. The spectrum is doubly degenerate in this case.

(iii) In the case of fixed boundary conditions, the CTM does not commute with Γ_{2N+1} and we again use the 'order-disorder' transformation in which Γ_1 is the parity operator.

The eigenvalue problem has two steps to its solution: first we find the operators $G_j^{\dagger}$, G_j and their associated eigenvalues λ_j, then we evaluate the maximal eigenvalue $\Lambda_{\max}$. Determination of P_{ij} for use in (3.9) is again an eigenvalue problem for a matrix of dimension $2N$, this time found by requiring

$$A(u)G_j^{\dagger}A^{-1}(u) = \lambda_j G_j^{\dagger} \tag{4.2}$$

which is a consequence of (4.1). $A(u)$ is the product of operators X_k whose effect on each generator Γ_j as a similarity transformation is to induce a linear transformation (automorphism of the algebra). The operators X_k are given in (3.6) as spinor rotation operators: $\exp(ix\Gamma_j\Gamma_{j+1})$. The basic relations from which everything follows are (Davies and Abraham 1987)

$$\exp(ix\Gamma_j\Gamma_k)\begin{bmatrix}\Gamma_j\\ \Gamma_k\end{bmatrix}\exp(-ix\Gamma_j\Gamma_k) = \begin{bmatrix}\cosh 2x & -i\sinh 2x\\ i\sinh 2x & \cosh 2x\end{bmatrix}\begin{bmatrix}\Gamma_j\\ \Gamma_k\end{bmatrix} \tag{4.3}$$

$$\exp(ix\Gamma_j\Gamma_k)\Gamma_l\exp(-ix\Gamma_j\Gamma_k) = \Gamma_l \qquad (l \neq j, k).$$

Consequently there is a complex $2N \times 2N$ matrix $\mathscr{X}_k(u)$ which represents the effect of X_k as a similarity transformation. Its definition is

$$X_k(u)\Gamma_i X_k(u)^{-1} = \sum_{j=1}^{2N} \mathscr{X}_k(u)_{ij}\Gamma_j \tag{4.4}$$

and from (3.6) and (4.3), we see it differs from the $2N \times 2N$ identity matrix only in one 2×2 block, between an adjacent pair of indices, where it has the entries

$$\mathscr{X}_k(u) = \begin{bmatrix}\sec 2u & -i\tan 2u\\ i\tan 2u & \sec 2u\end{bmatrix}. \tag{4.5}$$

This matrix is both complex Hermitian and orthogonal. Moreover its inverse is $\mathscr{X}_k(-u)$, which is the analogue of the inversion relation (2.15) for $X_k(u)$.

It follows that there is a complex $2N \times 2N$ matrix $\mathscr{A}(u)$ which represents the similarity transformations needed for the left-hand side of (4.2): namely

$$A(u)\Gamma_i A^{-1}(u) = \sum_{j=1}^{2N} \mathscr{A}(u)_{ij}\Gamma_j \tag{4.6}$$

where $\mathscr{A}(u)$ is the product of the appropriate matrices $\mathscr{X}_k(u)$. For free boundaries, (2.8) gives

$$\begin{aligned}\mathscr{A}(u) = {} & \mathscr{X}_{2N-1}\mathscr{X}_{2N-2}(\mathscr{X}_{2N-1}\mathscr{X}_{2N-3})(\mathscr{X}_{2N-2}\mathscr{X}_{2N-4})(\mathscr{X}_{2N-1}\mathscr{X}_{2N-3}\mathscr{X}_{2N-5})\\ & \ldots\mathscr{X}_2\ldots(\mathscr{X}_{2N-5}\mathscr{X}_{2N-3}\mathscr{X}_{2N-1})(\mathscr{X}_{2N-4}\mathscr{X}_{2N-2})(\mathscr{X}_{2N-3}\mathscr{X}_{2N-1})\mathscr{X}_{2N-2}\mathscr{X}_{2N-1}\end{aligned} \tag{4.7}$$

and there is a similar formula for fixed boundary conditions. In general we cannot give a closed formula for $\mathscr{A}(u)$ beyond this. However we can make a number of general observations which are of importance.

(i) $\mathscr{A}(u)$ is the product of Hermitian and complex orthogonal matrices and is invariant under reversal of the order of the factors $\mathscr{X}_k(u)$. Consequently $\mathscr{A}(u)$ is Hermitian and complex orthogonal; in particular the transpose and the complex conjugate of $\mathscr{A}(u)$ are both equal to its inverse.

(ii) The relation between $\mathscr{A}(u)$ and $\mathscr{H}$ is the same as between the normalised $A(u)$ and H: in the thermodynamic limit, $\mathscr{A}(u) = \exp(-u\mathscr{H})$ with $\mathscr{H} = \mathscr{A}'(0)$. Moreover, the inversion identity for $A(u)$ becomes $\mathscr{A}(u)^{-1} = \mathscr{A}(-u)$, without a normalisation factor.

(iii) The eigenvalues of $\mathscr{A}(u)$ are real and their product is unity for all complex u. For $u = 0$ all eigenvalues are unity: since $\mathscr{A}(u)$ is meromorphic in u, none of its eigenvalues may change sign, so they are positive for all complex u (away from poles of $\mathscr{A}(u)$).

(iv) The inverse of $\mathscr{A}(u)$ is its complex conjugate $\bar{\mathscr{A}}(u)$. Therefore, if v is an eigenvector of $\mathscr{A}(u)$ with eigenvalue λ, it is an eigenvector of $\bar{\mathscr{A}}(u) = \mathscr{A}(u)^{-1}$ with the eigenvalue λ^{-1} and the complex conjugate of v is a different eigenvector of $\bar{\mathscr{A}}(u)$ with eigenvalue λ. The last property follows from the fact that the eigenvectors are complex because $\mathscr{A}(u)$ is complex while λ is real. Thus the eigenvalues of $\mathscr{A}(u)$ must occur in real reciprocal pairs λ, λ^{-1} for which the corresponding eigenvectors are complex conjugate pairs. This corresponds to the pairing of eigenvalues of $\mathscr{H}$.

(v) Construct a unitary matrix Q which diagonalises $\mathscr{A}(u)$, arranged so that successive pairs of columns are eigenvectors belonging to eigenvalues λ_j, λ_j^{-1}, respectively, and with the λ_j ($\leqslant 1$) arranged in decreasing order of magnitude. Then the column of operators given by $\Sigma Q_{ij}\Gamma_j$ are just the required fermion operators $G_1^\dagger, G_1, G_2^\dagger, G_2, \ldots, G_N^\dagger, G_N$ and the generators Γ_i' constructed by using them in (3.8) define a real orthogonal transformation (3.9) because of the complex conjugate pairing of eigenvectors.

To complete the diagonalisation of $A(u)$ we must evaluate the maximal eigenvalue $\Lambda_{\max}$. This is done using the inversion identity (2.14), together with the operator (Davies 1988, equation (5.10))

$$R = (-1)^N \Gamma_2 \Gamma_4 \ldots \Gamma_{2N-2}\Gamma_{2N}. \tag{4.8}$$

R is an involution and it is also a unitary operator which commutes with $\Gamma_1, \Gamma_3, \ldots \Gamma_{2N-1}, \Gamma_{2N+1}$ and anticommutes with $\Gamma_2\Gamma_4 \ldots \Gamma_{2N-2}\Gamma_{2N}$. Its action of inverting the operators $X_k(u)$ may be seen from (3.6), from which the action on $A(u)$ follows:

$$X_k(-u) = RX_k(u)R^\dagger \qquad A(-u) = RA(u)R^\dagger. \tag{4.9}$$

Its action on the fermion operators $G_j^\dagger$, G_j is to interchange them with some change of phase: here we need only the simple result

$$RG_j^\dagger G_j R^\dagger = G_j G_j^\dagger. \tag{4.10}$$

Substituting (4.9), (4.1) and (4.10) into (4.6) (in that order) and using the canonical commutation relations for fermion operators ($G_j^\dagger G_j + G_j G_j^\dagger = 1$) gives

$$(2\cot 2u)^{N_1} = (\Lambda_{\max})^2 \prod_{j=1}^{N} \lambda_j \tag{4.11}$$

from which $\Lambda_{\max}$ may be determined once $\mathscr{A}(u)$ has been diagonalised.

The methods given for diagonalisation do not lead to closed form expressions for the spectrum of either H or $A(u)$. We know that the spectrum of $A(u)$ has a direct product form, with the structure given by the excitations and the overall scale set by $\Lambda_{\max}(u, N)$. For the purpose of this paper, in which interest is focused on the conformal properties of $A(u)$, we want to find the structure of the lower lying states, and the universal finite-size corrections to $\Lambda_{\max}(u, N)$. In general, all we know is that calculation of some property based on H will give the first term in a Taylor expansion, around $u = 0$, of the same property based on $A(u)$. Away from criticality, this information

would be sufficient because of the exponential relation between $A(u)$ and H. For the low lying states, this exponential property survives even at criticality. That is, for fixed u and n, we have the limiting behaviour

$$\lim_{N\to\infty} \frac{\lambda_n(u, N)}{\Lambda_{\max}(u, N)\exp[-u\varepsilon_n(u, N)]} = 1 \qquad (u, n \text{ fixed}). \tag{4.12}$$

This assertion depends on properties which cannot be proved analytically, so we report in an appendix numerical calculations which provide strong supporting evidence. The results of this paper, however, are based on analytic calculation alone. This is important as it should allow extension of the present methods from the Ising model to other solvable models for which there is no elegant algebraic formulation, but for which there are appropriate generators H with a quasiparticle spectrum.

5. Asymptotic analysis of generators

Truong and Peschel (1988) and Davies (1988) have shown that the low-lying eigenvalues of Ising and six-vertex CTMs are found from the zeros of Meixner polynomials (Chihara 1978), and that these zeros collapse like $1/\log N$ for large N. It is easy to show that the characteristic polynomials of $\mathcal{H}_0$, $\mathcal{H}_\pm$, of (3.13) are indeed Meixner polynomials. Let

$$\mathcal{M}_{ij} = i\delta_{i,j+1} + j\delta_{i+1,j} \tag{5.1}$$

be a square matrix of dimension n, which is the common part of $\mathcal{H}_0$ and $\mathcal{H}_\pm$ with the factor $\frac{1}{2}$ omitted for convenience. Then we may write a recursion relation for its characteristic polynomial $\mathcal{P}_n(\lambda)$ using Gaussian elimination on the tridiagonal matrix $(\lambda I - \mathcal{M})$ which completely determines $\mathcal{P}_n(\lambda)$ by

$$\begin{aligned} &\mathcal{P}_0(\lambda) = 1 \\ &\mathcal{P}_1(\lambda) = \lambda \\ &\mathcal{P}_{n+1}(\lambda) = \lambda\mathcal{P}_n(\lambda) - n^2\mathcal{P}_{n-1}(\lambda). \end{aligned} \tag{5.2}$$

These are the recursion relations for Meixner polynomials of the second kind (Chihara 1978). The eigenvalues of $\mathcal{H}_0$ are $\lambda/2$, where λ are the zeros of $\mathcal{P}_{2N-1}(\lambda)$ while the eigenvalues of $\mathcal{H}_\pm$ are similarly related to the zeros of $\mathcal{P}_{2N}(\lambda)$. The spectrum of excitations for the two different boundary conditions therefore interleave; also $\mathcal{H}_0$ has a total of two zero eigenvalues and $\mathcal{H}_\pm$ has none, since Meixner polynomials of odd (even) order are odd (even) functions. (Recall that $\mathcal{H}_0$ has a border of zeros which contribute one zero eigenvalue.) Both these properties are the same as in the non-critical case (Davies 1988).

Meixner polynomials have the integral representation (Truong and Peschel 1988a)

$$\mathcal{P}_n(\lambda) = \frac{n!\mathrm{i}^{-n}\cosh \pi\lambda/2}{\pi} \int_{-\infty}^{\infty} \frac{\tanh^n x \exp(\mathrm{i}\lambda x)}{\cosh x}\,\mathrm{d}x. \tag{5.3}$$

The asymptotic form for large n is found using the method of steepest descents. Define the phase function $\phi(x, \lambda)$ by

$$\phi(x, \lambda) = \mathrm{i}\lambda x + n\log\sinh x - (n+1)\log\cosh x. \tag{5.4}$$

The points of stationary phase in the complex x plane are determined by

$$0 = \phi'(x, \lambda) = i\lambda + n \coth x - (n+1) \tanh x \tag{5.5}$$

which determines two stationary points $x_\pm$ via

$$\tanh x_\pm = \frac{i\lambda \pm [4n(n+1) - \lambda^2]^{1/2}}{2(n+1)}. \tag{5.6}$$

The lowest-order steepest descent approximation is

$$\mathscr{P}_n(\lambda) \approx \frac{n!\, i^{-n} \cosh \pi\lambda/2}{\pi} [\exp \psi(x_-, \lambda) + \exp \psi(x_+, \lambda)] \tag{5.7}$$

where

$$\psi(x, \lambda) = \phi(x, \lambda) - \tfrac{1}{2} \log[-2\phi''(x, \lambda)]. \tag{5.8}$$

In order to get a zero, which is our primary interest, the two terms in (5.7) must cancel. Remember also that λ is real. So we need that $\mathrm{Re}[\psi(x_-, \lambda)] = \mathrm{Re}[\psi(x_+, \lambda)]$ and $\mathrm{Im}[\psi(x_-, \lambda)] - \mathrm{Im}[\psi(x_+, \lambda)] = \pi \pmod{2\pi}$. Let us consider first real λ in the interval $(0, [4n(n+1)]^{1/2})$. Then it may be shown that the function $\mathrm{Im}[\psi(x_+, \lambda)]$ increases monotonically from the limit $\mathrm{Im}[\psi(x_+, 0+)] = 0$ to $\mathrm{Im}[\psi(x_+, [4n(n+1)]^{1/2} - 0)] = (n+\frac{1}{2})\pi/2$. Similarly the function $\mathrm{Im}[\psi(x_-, \lambda)]$ decreases monotonically from the limit $\mathrm{Im}[\psi(x_-, 0+)] = n\pi$ to $\mathrm{Im}[\psi(x_-, [4n(n+1)]^{1/2} - 0)] = (n-\frac{1}{2})\pi/2$. The stationary points are symmetrically placed about the imaginary axis, with $\mathrm{Im}[x_\pm] > 0$ and it follows that $\mathrm{Re}[\psi(x_-, \lambda)] = \mathrm{Re}[\psi(x_+, \lambda)]$ for all λ in this interval. Consequently the condition for a zero is satisfied the correct number of times: that is, $[n/2]$ (integer part of $n/2$). Also there is a zero at $\lambda = 0$ for odd n because the limit $\mathrm{Im}[\psi(x_+, 0+)] - \mathrm{Im}[\psi(x_-, 0-)] = \pi \pmod{2\pi}$ when n is odd. For $\lambda = [4n(n+1)]^{1/2}$, the stationary points coalesce on the imaginary axis and change their nature. For $\lambda > [4n(n+1)]^{1/2}$ there are no more zeros. Steepest descents generates an asymptotic expansion, and in principle we may take this to whatever order is needed. At any order, however, we recover the same structure with some phase function $\psi(x, \lambda)$.

We introduce some new variables before giving the formulae for the zeros, writing

$$\alpha = \lambda/[4n(n+1)]^{1/2} \qquad f(\alpha) = 2\,\mathrm{Im}[\psi(x_+, \lambda)]/\pi n. \tag{5.9}$$

The positive zeros lie in the range $0 < \alpha < 1$ and are given by the implicit equation

$$\begin{aligned} f(\alpha_j) &= (2j-1)/n \qquad (j = 1, \ldots, n/2, \text{even } n) \\ f(\alpha_j) &= 2j/n \qquad\quad (j = 1, \ldots, [n/2], \text{odd } n). \end{aligned} \tag{5.10}$$

The asymptotic values for the low-lying part of the spectrum is contained in the behaviour of $f(\alpha)$ in some neighbourhood of $\alpha = 0$, which remains unchanged as $n \to \infty$. From the definitions of $f(\alpha)$ and $\mathrm{Im}[\psi(x_+, \lambda)]$ we find that for small α

$$f(\alpha) \approx -(2\alpha/\pi) \log(\alpha/2) \tag{5.11}$$

from which it follows that, for fixed λ and large n, the Meixner polynomials have the asymptotic form

$$\begin{aligned} \mathscr{P}_n(\lambda) &\approx \sin[(\lambda/2) \log n] \qquad (\text{even } n) \\ &\approx \cos[(\lambda/2) \log n] \qquad (\text{odd } n). \end{aligned} \tag{5.12}$$

The excitation energies of low-lying states (recall that energies are normalised per radian and also that $\varepsilon_j = \lambda_j/2$) are given asymptotically by

$$\varepsilon_j \approx \pi j/\log N \tag{5.13}$$

for $\mathscr{H}_0$ and

$$\varepsilon_j \approx \pi(j-\tfrac{1}{2})/\log N \tag{5.14}$$

for $\mathscr{H}_\pm$. Thus the low-lying part of the spectrum collapses for large N. In using these results it is important to notice that the higher part of the spectrum certainly does not collapse. This is seen by writing

$$\mathscr{P}_n(\lambda) = x^n + a_2 x^{n-2} + \ldots . \tag{5.15}$$

The coefficient a_2 is the sum of the squares of the eigenvalues, and the recursion relation for $\mathscr{P}_n(\lambda)$ gives

$$a_2 = \sum_{j=1}^{n-1} j^2 \qquad (n>1) \tag{5.16}$$

which prevents the collapse of all but the bottom of the spectrum.

Now we need to calculate the maximal eigenvalue $\Lambda_{\max}$, at least for small u. We write (4.11) as

$$\log \Lambda_{\max} = \tfrac{1}{2} N_1 \log 2 \cot 2u + 2u \sum_{j=1}^{N} \varepsilon_j . \tag{5.17}$$

We must obtain an asymptotic expansion for the sum for large N. Consider first the case of free boundaries. The Meixner polynomial is of degree $2N-1$, and there are $N-1$ excitations ε_j determined by the implicit equation (5.10a). Thus

$$\sum_{j=1}^{N-1} \varepsilon_j = [4N(N+1)]^{1/2} \sum_{j=1}^{N-1} f^{-1}(j/N). \tag{5.18}$$

Here f^{-1} stands for the inverse function; its values at an equally spaced set of points determines the eigenvalues. The leading approximation to (5.18) is evidently an integral of $f^{-1}(x)$ over the range $[0, 1]$. The asymptotic expansion is the error expansion of a quadrature rule for such an integral, and the logarithmic singularity of the integrand is responsible for the corrections we seek. Therefore we write (5.18) as

$$\sum_{j=1}^{N-1} \varepsilon_j = N[4N(N+1)]^{1/2} T_{\text{end}}^{[0-1]} f^{-1}(x) \tag{5.19}$$

where the notation indicates the endpoint trapezoidal quadrature rule for the integral over the interval $[0, 1]$. Asymptotic expansions for the error of general quadrature rules, when the integrand has end point singularities, were developed by Lyness and Ninham (1966). They give fomulae for an endpoint singularity of the type $x^\beta h(x)$, where $h(x)$ is analytic, from which the formulae for functions with a logarithmic singularity may be obtained by differentiation or integration with respect to β. For the endpoint rule, the formula is

$$T_{\text{end}}^{[0-1]} x^\beta h(x) - \int_0^1 x^\beta h(x)\,\mathrm{d}x \approx \sum_{k\geqslant 0} \frac{a_k}{N^{\beta+k+1}} + \frac{b_k}{N^{k+1}}. \tag{5.20}$$

Near $\alpha = 0$, the inverse function $f^{-1}(x)$ has the form

$$f^{-1}(x) \approx -\frac{\pi x}{2 \log x} \tag{5.21}$$

so, in order to use (5.20), we must integrate both sides with respect to β and then set $\beta = 1$. The leading logarithmic correction ($k = 0$) is a term $1/N^2 \log N$, and the N^2 factor cancels the prefactor in (5.19). The coefficient a_0 may be read off from Lyness and Ninham (1966): it is

$$a_0 = \zeta(-1)h(0) = -h(0)/12 \tag{5.22}$$

where $\zeta(s)$ is the Riemann zeta function. Putting this all together, we find, for free boundary conditions,

$$\log \Lambda_{\max} \approx A_0 N^2 + B_0 N + C_0 + \pi/24 \log N. \tag{5.23}$$

The convention here is that the geometric factor $4u$ has been dropped, as with the excitation energies ε_j. The coefficients A_0, B_0, C_0, will only be correct for infinitesimal u, but the next term is universal. In fact, this might be argued from its very origin as a factor which depends only on the nature of the singularity near zero α. The same derivation may be repeated for fixed boundary conditions. This time we need the asymptotic expansion for the sum

$$\begin{aligned} \sum_{j=1}^{N-1} \varepsilon_j &= [4N(N+1)]^{1/2} \sum_{j=1}^{N-1} f^{-1}((j-\tfrac{1}{2})/N) \\ &= N[4N(N+1)]^{1/2}\, T_{\text{mid}}^{[0-1]} f^{-1}(x) \end{aligned} \tag{5.24}$$

with the midpoint trapezoidal quadrature rule. This time we have

$$a_0 = (2^{-1} - 1)\zeta(-1)h(0) = h(0)/24 \tag{5.25}$$

giving

$$\log \Lambda_{\max} \approx A_{\pm} N^2 + B_{\pm} N + C_{\pm} - \pi/48 \log N. \tag{5.26}$$

6. Virasoro characters

CTMs are themselves partition functions of a quadrant of the lattice. According to the theory of conformal invariance, partition functions for a large but finite critical system may be factored into the product of a non-universal part, which contains the information about bulk free energy, etc, and a universal part which is manifested in the finite-size corrections. For CTMs, Peschel and Truong (1987) have shown that the leading finite-size corrections involve $\log N$; such behaviour is exactly what we have seen in section 4 in our asymptotic analysis.

First we make some general observations which are independent of the boundary conditions. There are 2^N eigenvalues of the CTM and (4.1) and (4.3) shows that we may write them as

$$\Lambda = \Lambda_{\max} \lambda_1^{n_1} \lambda_2^{n_2} \ldots \lambda_N^{n_N} \tag{6.1}$$

where the occupation numbers n_j take the values 0 or 1. The corresponding eigenstate is generated from the maximal eigenstate by application of the creation operators $G_j^{\dagger}$ for each non-zero n_j. The parity of the state—in the fermion language—depends on whether there are an even or odd number of excitations. The collection of all eigenvalues may be represented by the generating functions

$$\chi_{\pm}(\lambda_1, \lambda_2, \ldots, \lambda_N) = \Lambda_{\max} \prod_{j=1}^{N} (1 \pm \lambda_j). \tag{6.2}$$

By this we mean that the 2^N terms in this product are in one-to-one correspondence with the eigenvalues given in (6.1). Moreover, the generating function χ_- also characterises the parity of the eigenstates correctly as the sign of the product. If now we represent $\Lambda_{\max}$ as a product of non-universal and universal factors Λ_{nu} and Λ_0 ($\Lambda_\pm$):

$$\Lambda_{\max} = \Lambda_{nu}\Lambda_0 \qquad (\Lambda_{nu}\Lambda_\pm) \tag{6.3}$$

then we know the dependence of Λ_0 ($\Lambda_\pm$) on $\log N$ from the analysis of section 4, and we should discard the non-universal part when considering the theory of conformal invariance.

Now we may construct the generating functions for the universal part of the eigenvalues in the limit of large N. First, for free boundary conditions the CTM has two identical blocks, labelled by the value of the centre spin. Equations (5.13) and (5.23) show that the asymptotic values which we need are

$$\Lambda_0 \approx q^{1/24} \qquad \lambda_j \approx q^j \tag{6.4}$$

where

$$q = \exp(-\pi/\log N). \tag{6.5}$$

Substituting this into (6.3) and dropping the non-universal contribution, we find that asymptotically the generator of the eigenvalue spectrum is the Virasoro character:

$$\begin{aligned}\chi_{1/16}(q) &= q^{-1/48+1/16}\prod_{j=1}^{\infty}(1+q^j)\\ &= q^{+1/24}(1+q+q^2+2q^3+2q^4+\dots).\end{aligned} \tag{6.6}$$

For fixed boundary conditions, there are the two blocks of the CTM for which the centre spin is aligned (opposite) to the boundary spins. Expressed in fermion operators, these are the even (odd) parity solutions. The generating functions for the spectrum in the two cases are

$$\chi_+(\lambda_1, \lambda_2, \dots, \lambda_N) \pm \chi_-(\lambda_1, \lambda_2, \dots, \lambda_N) \tag{6.7}$$

respectively. In either case, the relevant asymptotic values, (5.14) and (5.26) are

$$\Lambda_\pm \approx q^{-1/48} \qquad \lambda_j \approx q^{(j-1/2)}. \tag{6.8}$$

Thus the asymptotic limit of the generators (6.8) are expressed as the linear combinations

$$\chi_0(q) \pm \chi_{1/2}(q) \tag{6.9}$$

of the two Virasoro characters $\chi_0(q), \chi_{1/2}(q)$, where

$$\begin{aligned}\chi_0(q) &= q^{-1/48}\left(\prod_{j=1}^{\infty}(1+q^{j-1/2}) + \prod_{j=1}^{\infty}(1-q^{j-1/2})\right)\\ &= q^{-1/48}(1+q^2+q^3+2q^4+2q^5+\dots)\end{aligned} \tag{6.10}$$

$$\begin{aligned}\chi_{1/2}(q) &= q^{-1/48}\left(\prod_{j=1}^{\infty}(1+q^{j-1/2}) - \prod_{j=1}^{\infty}(1-q^{j-1/2})\right)\\ &= q^{-1/48}\{q^{1/2}+q^{3/2}+q^{5/2}+q^{7/2}+2q^{9/2}+\dots\}.\end{aligned} \tag{6.11}$$

7. Conclusion

There is increasing evidence of a close relation between CTMs and the Virasoro algebra. The Virasoro characters $\chi_\Delta(q)$ occur in the spectrum of CTMs in two distinct instances. First, for solvable models such as the *A-D-E* models associated with classical Lie algebras, they appear in the exact expressions obtained for the non-critical local state probabilities in the thermodynamic limit. In this case the modular parameter q is related to the deviation from the critical temperature. Second, the Virasoro characters appear in the spectrum of CTMs at criticality. This has been established for the Ising model in this paper and is expected to hold more generally for the *A-D-E* models. In this instance the modular parameter q is related to the size of the finite geometry. From the point of view of scaling, however, there should be just one scale in the theory away from the critical point of the bulk system, given by the correlation length ξ. Indeed, as pointed out by Saleur and Bauer (1989), both cases are correctly described by the single formula

$$q = \exp(-\pi\theta/\log \xi) \tag{7.1}$$

where the opening or effective angle θ is 2π for a plane.

For the sake of definiteness, let us consider the L-state *ABF* models so that $L=3$ is the Ising model. Then in the ordered regime III (Andrews *et al* 1984) the modular parameter of the Virasoro characters is given by

$$q = \exp[-4\pi\theta/(L+1)\varepsilon] \tag{7.2}$$

where $p=\exp(-\varepsilon)$ is the temperature variable. The effective angle θ of a corner is related (Barber *et al* 1984, Kim and Pearce 1987) to the spectral parameter u of the corner by $\theta=(L+1)u$. If we use the result

$$\xi \approx p^{-\nu} \qquad \nu=(L+1)/4 \tag{7.3}$$

to eliminate ε in (7.2) in favour of ξ we see that this modular parameter agrees with (7.1). Similarly, returning to the result (6.5), which is valid for a finite system at criticality, and using the finite-size scaling hypothesis

$$\xi \approx N \qquad N \text{ large} \tag{7.4}$$

also gives agreement with (7.1) for the angle θ equal to one radian.

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Appendix

For the purpose of this paper, in which interest is focused on conformal properties, we have only used the structure of the lower lying states of $A(u)$. All results needed for this purpose have been obtained directly from H using analytic methods. In this appendix we provide numerical evidence that using the exponential relation between the two does give the correct description for the low-lying states.

The matrix $\mathscr{A}(u)$ is of dimension $2N$ only, a simplification which allows easy numerical calculation to quite large values of N. We have performed calculations for both free and fixed boundary conditions. The relevant formulae for building up $\mathscr{A}(u)$ are given in section 4. It is evident that the computation of $\mathscr{A}(u)$ for $N+1$ sites consists mainly of a repetition of the steps necessary for N sites. Since the purpose of the calculation is to study the dependence of eigenvalues on N, we wrote a FORTRAN program in which N was increased in steps of one, and at each step the eigenvalues $\lambda_n(u, N)$ of $\mathscr{A}(u)$ were calculated. Also at each step, the matrix $\mathscr{H}$, which represents the generator, was constructed and its eigenvalues ε_n found. Finally, for fixed u and n, the ratio of $\lambda_n(u, N)/\Lambda_{\max} \exp(-u\varepsilon_n)$ was calculated as a function of N. The practical limit on these calculations is the (numerical) condition of $\mathscr{A}(u)$, as indicated by the ratio of the maximum to minimum eigenvalues. Together with the fact that the asymptotics involve the logarithm of N this makes it impossible to pursue the calculations to very large size for $\mathscr{A}(u)$. In all cases our calculations indicate, to the double precision employed, that the ratio (4.12) converges slowly to unity. We have done similar calculations for fixed boundaries also, with the same result. Results from a representative calculation, for free boundary conditions, and with $u = 0.05$, are shown in table 1.

Table 1. Numerical values of ratio of eigenvalues of $\mathscr{A}_0$ and $\mathscr{H}_0$, for the two lowest states, with $u = 0.05$—see (4.12). Also shown is the ratio of the minimum and maximum eigenvalues of $\mathscr{A}_0$ (condition number).

N	Ratio $\lambda_{\min}/\lambda_{\max}$	Ratio (4.12) for $n=1$	Ratio (4.12) for $n=2$
6	5.429	1.000 190 262 392 37	1.000 677 955 083 08
8	11.07	1.000 175 749 614 29	1.000 506 657 673 20
12	47.73	1.000 159 161 331 94	1.000 378 796 066 93
16	212.0	1.000 149 450 959 32	1.000 322 339 556 51
24	4381.0	1.000 137 923 859 16	1.000 267 402 827 52
32	9.3852E+04	1.000 130 945 042 57	1.000 239 163 699 45
48	4.5661E+07	1.000 122 415 052 30	1.000 208 808 773 50
64	2.3234E+10	1.000 117 106 225 22	1.000 191 899 661 68
96	5.8839E+15	1.000 110 453 009 85	1.000 172 566 788 91
128	1.8569E+16	1.000 106 215 066 36	1.000 161 217 720 28

To illustrate further how painfully slow numerical convergence is, we have given some further calculations, for $\mathscr{H}_0$, table 2. The (numerical) condition of $\mathscr{H}_0$, unlike that of $\mathscr{A}(u)$, does not deteriorate rapidly for large N. We have shown the two lowest eigenvalues up to $N = 2048$, together with their asymptotic values as given in (5.13) and as calculated from the solution of the transcendental equation (5.10) using the asymptotic form (5.11). The second method corresponds to the way in which the central charge was recovered in section 5. We have also shown the ratio of the lowest two eigenvalues, known to have the asymptotic value of 2. The slow convergence is evident from the numbers.

As a further cross-check on the coding, we also made direct numerical calculations for the CTM $A(u)$. We calculated the spectra of $A(u)$ for a number of values of u and for $N = 1, 2, \ldots, 8$ (a maximum of 256 eigenvalues) and compared with the indirect calculation using $\mathscr{A}(u)$, followed by the inversion identity (4.11) to obtain $\Lambda_{\max}$. In all cases there was complete agreement (to the double precision employed).

Table 2. Numerical values for lowest eigenvalues of $\mathcal{H}_0$.

N	Ratio of ε_1 to value given by (5.13)	Ratio of ε_1 to solution of (5.10) and (5.11)	Ratio of ε_1 to ε_2
16	0.498 320 485 829	0.797 642 292 638	2.289 458 819 602
24	0.527 146 259 445	0.824 832 990 476	2.248 035 227 752
32	0.545 559 595 235	0.840 935 043 215	2.224 201 202 420
48	0.569 101 319 323	0.860 090 116 066	2.196 267 104 041
64	0.584 325 230 480	0.871 648 891 959	2.179 555 982 510
96	0.603 991 155 770	0.885 670 140 023	2.159 389 906 076
128	0.616 825 807 531	0.894 301 092 799	2.147 032 687 113
192	0.633 538 133 611	0.904 970 236 915	2.131 834 529 442
256	0.644 524 516 277	0.911 657 375 587	2.122 369 271 208
384	0.658 922 676 208	0.920 059 121 055	2.110 571 338 427
512	0.668 444 158 683	0.925 404 484 992	2.103 137 029 109
768	0.680 989 546 751	0.932 208 919 175	2.093 777 687 924
1024	0.689 327 258 801	0.936 589 440 295	2.087 826 986 937
1536	0.700 362 965 362	0.942 222 833 827	2.080 276 708 844
2048	0.707 728 637 036	0.945 882 702 579	2.075 441 839 771

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